ELLIPTIC WEYL GROUP ELEMENTS AND UNIPOTENT ISOMETRIES WITH $p = 2$

George Lusztig and Ting Xue

Introduction

0.1. Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$. Let $\underline{\underline{G}}$ be the set of unipotent conjugacy classes in G. Let $\underline{\mathbf{W}}$ be the set of conjugacy classes in the Weyl group $\mathbf{W}$ of G. For $w \in \mathbf{W}$ and $\gamma \in \underline{\underline{G}}$ let $\mathfrak{B}_w^\gamma$ be the variety of all pairs (g, B) where $g \in \gamma$ and B is a Borel subgroup of G such that B and gBg^{-1} are in relative position w. For $C \in \underline{\mathbf{W}}$ and $\gamma \in \underline{\underline{G}}$ we write $C \dashv \gamma$ when for some (or equivalently any) element w of minimal length in C we have $\mathfrak{B}_w^\gamma \neq \emptyset$. In [L1, 4.5] a natural surjective map $\Phi : \underline{\mathbf{W}} \to \underline{\underline{G}}$ was defined. When p is not a bad prime for G, the map Φ can be characterized in terms of the relation $C \dashv \gamma$ as follows (see [L1, 0.4]):

(a) *If $C \in \underline{\mathbf{W}}$ then $\Phi(C)$ is the unique unipotent class of G such that $C \dashv \Phi(C)$ and such that if $\gamma' \in \underline{\underline{G}}$ satisfies $C \dashv \gamma'$ then $\Phi(C)$ is contained in the closure of γ'.*

If p is a bad prime for G then the definition of the map Φ given in [L1] is less direct; one first defines Φ on elliptic conjugacy classes by making use of the analogous map in characteristic 0 and then one extends the map in a standard way to nonelliptic classes. It would be desirable to establish property (a) also in bad characteristic. To do this it is enough to establish (a) in the case where C is elliptic (see the argument in [L1, 1.1].) One can also easily reduce the general case to the case where G is almost simple; moreover it is enough to consider a single G in each isogeny class. The fact that (a) holds for C elliptic with G almost simple of exceptional type (with p a bad prime) was pointed out in [L2, 4.8(a)]. It remains then to establish (a) for C elliptic in the case where G is a symplectic or special orthogonal group and $p = 2$. This is achieved in the present paper. In fact, Theorem 1.3 establishes (a) with C elliptic in the case where G is $Sp_{2n}(\mathbf{k})$ or $SO_{2n}(\mathbf{k})$ ($p = 2$); then (a) for $G = SO_{2n+1}(\mathbf{k})$ ($p = 2$) follows from the analogous result for $Sp_{2n}(\mathbf{k})$ using the exceptional isogeny $SO_{2n+1}(\mathbf{k}) \to Sp_{2n}(\mathbf{k})$. Thus the results of this paper establish (a) for any G without restriction on p.

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0.2. If $w \in \mathbf{W}$ and $\gamma \in \underline{\underline{G}}$ then G_{ad} (the adjoint group of G) acts on $\mathfrak{B}_w^\gamma$ by $x : (g, B) \mapsto (xgx^{-1}, xBx^{-1})$. Let $C \in \underline{\mathbf{W}}$ be elliptic. Let $\gamma = \Phi(C)$. The following result is proved in [L2, 0.2].

(a) *For any $w \in C$ of minimal length, $\mathfrak{B}_w^\gamma$ is a single G_{ad}-orbit.*

The following converse of (a) appeared in [L2, 3.3(a)] in the case where p is not a bad prime for G and in the case where G is almost simple of exceptional type and p is a bad prime for G (see also [L1, 5.8(c)]):

(b) *Let $\gamma' \in \underline{\underline{G}}$. If $C \dashv \gamma'$ and $\gamma' \neq \Phi(C)$ then for any $w \in C$ of minimal length, $\mathfrak{B}_w^\gamma$ is a union of infinitely many G_{ad}-orbits.*

Using 0.1(a) we see as in the proof of [L1, 5.8(b)] that (b) holds for any G without restriction on p. Namely, from [L1, 5.7(ii)] we see that $\mathfrak{B}_w^{\gamma'}$ has pure dimension equal to $\dim \gamma' + l(w)$ where $l(w)$ is the length of w and $\mathfrak{B}_w^\gamma$ has pure dimension equal to $\dim \gamma + l(w)$. Also by [L1, 5.2] the action of G_{ad} on $\mathfrak{B}_w^{\gamma'}$ or $\mathfrak{B}_w^\gamma$ has finite isotropy groups. Thus, $\dim \mathfrak{B}_w^\gamma = \dim G_{ad}$ (see (a)) and to prove (b) it is enough to show that $\dim \mathfrak{B}_w^{\gamma'} > \dim G_{ad}$ or equivalently that $\dim \gamma' + l(w) > \dim \gamma + l(w)$ or that $\dim \gamma' > \dim \gamma$. But from 0.1(a) we see that γ is contained in the closure of γ'; since $\gamma \neq \gamma'$ it follows that $\dim \gamma' > \dim \gamma$, as required.

Note that (a),(b) provide, in the case where C is elliptic, another characterization of $\Phi(C)$ which does not rely on the partial order on $\underline{\underline{G}}$.

1. The main results

1.1. In this section we assume that $p = 2$. Let V be a $\mathbf{k}$-vector space of finite dimension $\mathbf{n} = 2n \geq 4$ with a fixed nondegenerate symplectic form $(,) : V \times V \to \mathbf{k}$ and a fixed quadratic form $Q : V \to \mathbf{k}$ such that (i) or (ii) below holds:

(i) $Q = 0$;

(ii) $Q \neq 0$, $(x, y) = Q(x + y) - Q(x) - Q(y)$ for $x, y \in V$.

Let $Is(V)$ be the group consisting of all $g \in GL(V)$ such that $(gx, gy) = (x, y)$ for all $x, y \in V$ and $Q(gx) = Q(x)$ for all $x \in V$ (a closed subgroup of $GL(V)$). Let G be the identity component of $Is(V)$. Let $\mathcal{F}$ be the set of all sequences $V_* = (0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{\mathbf{n}} = V)$ of subspaces of V such that $\dim V_i = i$ for $i \in [0, \mathbf{n}]$, $Q|_{V_i} = 0$ and $V_i^\perp = V_{\mathbf{n}-i}$ for all $i \in [0, n]$. Here, for any subspace V' of V we set $V'^\perp = \{x \in V; (x, V') = 0\}$.

1.2. Let $p_1 \geq p_2 \geq \cdots \geq p_\sigma$ be a sequence in $\mathbf{Z}_{>0}$ such that $p_1 + p_2 + \cdots + p_\sigma = n$. (In the case where $Q \neq 0$ we assume that σ is even.) For any $r \in [1, \sigma]$ we set $p_{\leq r} = \sum_{r' \in [1,r]} p_{r'}$, $p_{<r} = \sum_{r' \in [1,r-1]} p_{r'}$. We fix $(V_*, V'_*) \in \mathcal{F} \times \mathcal{F}$ such that for any $r \in [1, \sigma]$ we have

(a) $\dim(V'_{p_{<r}+i} \cap V_{p_{<r}+i}) = p_{<r} + i - r, \quad \dim(V'_{p_{<r}+i} \cap V_{p_{<r}+i+1}) = p_{<r} + i - r + 1$

if $i \in [1, p_r - 1]$;

(b) $\quad \dim(V'_{p_{\leq r}} \cap V_{\mathbf{n}-p_{<r}-1}) = p_{\leq r} - r, \quad \dim(V'_{p_{\leq r}} \cap V_{\mathbf{n}-p_{<r}}) = p_{\leq r} - r + 1.$

(Such (V_*,V'_*) exists and is unique up to conjugation by $Is(V)$.) A unipotent class γ in G is said to be adapted to (V_*,V'_*) if for some $g\in\gamma$ we have $gV_i=V'_i$ for all i.

There is a unique unipotent conjugacy class γ in G such that γ is adapted to (V_*,V'_*) and some/any element of γ has Jordan blocks of sizes $2p_1,2p_2,\dots,2p_\sigma$. (The existence of γ is proved in [L1, 2.6, 2.12]; the uniqueness follows from the proof of [L1, 4.6].)

Theorem 1.3. *Let γ' be a unipotent conjugacy class in G which is adapted to (V_*,V'_*). Then γ is contained in the closure of γ' in G.*

The proof is given in 1.5-1.8 (when $Q=0$) and in 1.9 (when $Q\ne 0$).

1.4. Let $\mathcal{T}$ be the set of sequences $c_*=(c_1\ge c_2\ge c_3\ge\dots)$ in $\mathbf{N}$ such that $c_m=0$ for $m\gg 0$ and $c_1+c_2+\dots=\mathbf{n}$. For $c_*\in\mathcal{T}$ we define $c^*_*=(c^*_1\ge c^*_2\ge c^*_3\ge\dots)\in\mathcal{T}$ by $c^*_i=|\{j\ge 1;c_j\ge i\}|$ and we set $\mu_i(c_*)=|\{j\ge 1;c_j=i\}|$ $(i\ge 1)$; thus we have

(a) $$\mu_i(c_*)=c^*_i-c^*_{i+1}.$$

For $i,j\ge 1$ we have

(b) $$i\le c_j \text{ iff } j\le c^*_i.$$

For $c_*\in\mathcal{T}$ and $i\ge 1$ we have

(c) $\sum_{j\in[1,c^*_i]}(c_j-i)+\sum_{j\in[1,i]}c^*_j=\mathbf{n}$.

Indeed the left hand side is

$$\begin{aligned}&\sum_{j\ge1;i\le c_j}(c_j-i)+\sum_{j\in[1,i],k\ge1;c_k\ge j}1=\sum_{j\ge1;i\le c_j}(c_j-i)+\sum_{k\ge1}\min(i,c_k)\\&=\sum_{j\ge1;i\le c_j}(c_j-i)+\sum_{k\ge1;i\le c_k}i+\sum_{k\ge1;i>c_k}c_k\\&=\sum_{j\ge1;i\le c_j}c_j+\sum_{k\ge1;i>c_k}c_k=\sum_{j\ge1}c_j=\mathbf{n}.\end{aligned}$$

For $c_*,c'_*\in\mathcal{T}$ and $i\ge 1$:

(d) *we have* $\sum_{j\in[1,i]}c^*_j=\sum_{j\in[1,i]}{c'}^*_j$ *iff* $\sum_{j\in[1,c^*_i]}(c_j-i)=\sum_{j\in[1,{c'}^*_i]}(c'_j-i)$*;*
we have $\sum_{j\in[1,i]}c^*_j\ge\sum_{j\in[1,i]}{c'}^*_j$ *iff* $\sum_{j\in[1,c^*_i]}(c_j-i)\le\sum_{j\in[1,{c'}^*_i]}(c'_j-i)$.

This follows from (c) and the analogous equality for c'_*.

For $c_*,c'_*\in\mathcal{T}$ we say that $c_*\le c'_*$ if the following (equivalent) conditions are satisfied:

(i) $\sum_{j\in[1,i]}c_j\le\sum_{j\in[1,i]}c'_j$ for any $i\ge 1$;

(ii) $\sum_{j\in[1,i]}c^*_j\ge\sum_{j\in[1,i]}{c'}^*_j$ for any $i\ge 1$.

We show:

(e) *Let* $c_*, c'_* \in \mathcal{T}$ *and* $i \ge 1$ *be such that* $c_* \le c'_*$, $\sum_{j\in[1,i]} c^*_j = \sum_{j\in[1,i]} c'^*_j$ *and* $\mu_i(c_*) > 0$. *Then* $c^*_i \le c'^*_i$ *and* $\mu_i(c'_*) > 0$.
We set $m = c^*_i, m' = c'^*_i$. From $c_* \le c'_*$ we deduce $\sum_{j\in[1,i-1]} c^*_j \ge \sum_{j\in[1,i-1]} c'^*_j$ (if $i = 1$ both sums are zero); using the equality $\sum_{j\in[1,i]} c^*_j = \sum_{j\in[1,i]} c'^*_j$ we deduce $c^*_i \le c'^*_i$ that is, $m \le m'$. From (d) we have
$\sum_{j\in[1,m]}(c_j - i) = \sum_{j\in[1,m']}(c'_j - i)$.
Hence

$$\sum_{j\in[1,m]} c_j = \sum_{j\in[1,m']} c'_j + (m-m')i$$

$$= \sum_{j\in[1,m]} c'_j + \sum_{j\in[m+1,m']} (c'_j - i) \ge \sum_{j\in[1,m]} c'_j \ge \sum_{j\in[1,m]} c_j; \tag{f}$$

we have used $c_* \le c'_*$ and that for $j \in [m+1, m']$ we have $i \le c'_j$ (since $j \le c'^*_i$, see (b)). It follows that the inequalities in (f) are equalities hence $c'_j = i$ for $j \in [m+1, m']$. Thus $\mu_i(c'_*) \ge m - m'$. This completes the proof of (e) in the case where $m > m'$. Now assume that $m = m'$. From $c_* \le c'_*$ we have $\sum_{j\in[1,m-1]} c_j \le \sum_{j\in[1,m-1]} c'_j$. Using this and (d) we see that

$$\sum_{j\in[1,m]} (c_j - i) = \sum_{j\in[1,m]} (c'_j - i) \ge \sum_{j\in[1,m-1]} (c_j - i) + c'_m - i$$

hence $c_m - i \ge c'_m - i$. From $\mu_i(c_*) > 0$ and $c^*_i = m$ we deduce that $c_m = i$. (Indeed by 1.4(b) we have $i \le c_m$; if $i < c_m$ then $i+1 \le c_m$ and by 1.4(b) we have $m \le c^*_{i+1} \le c^*_i = m$ hence $c^*_{i+1} = c^*_i$ and $\mu_i(c_*) = 0$, contradiction.) Hence $c'_m \le i$. Since $c'^*_i = m$ we have also $i \le c'_m$ (see (b)) hence $c'_m = i$. Thus $\mu_i(c'_*) > 0$. This completes the proof of (e).

1.5. In this subsection (and until the end of 1.8) we assume that $Q = 0$. In this case we write $Sp(V)$ instead of $Is(V) = G$. Let u be a unipotent element of $Sp(V)$. We associate to u the sequence $c_* \in \mathcal{T}$ whose nonzero terms are the sizes of the Jordan blocks of u. We must have $\mu_i(c_*)$ =even for any odd i. We also associate to u a map $\epsilon_u : \{i \in 2\mathbf{N}; i \ne 0, \mu_i(c_*) > 0\} \to \{0, 1\}$ as follows: $\epsilon_u(i) = 0$ if $((u-1)^{i-1}(x), x) = 0$ for all $x \in \ker(u-1)^i : V \to V$ and $\epsilon_u(i) = 1$ otherwise; we have automatically $\epsilon_u(i) = 1$ if $\mu_i(c_*)$ is odd. Now $u \mapsto (c_*, \epsilon_u)$ defines a bijection between the set of conjugacy classes of unipotent elements in $Sp(V)$ and the set $\mathfrak{S}$ consisting of all pairs (c_*, ϵ) where $c_* \in \mathcal{T}$ is such that $\mu_i(c_*)$ =even for any odd i and $\epsilon : \{i \in 2\mathbf{N}; i \ne 0, \mu_i(c_*) > 0\} \to \{0, 1\}$ is a function such that $\epsilon(i) = 1$ if $\mu_i(c_*)$ is odd. (See [S, I,2.6]). We denote by $\gamma_{c_*,\epsilon}$ the unipotent class corresponding to $(c_*, \epsilon) \in \mathfrak{S}$. For $(c_*, \epsilon) \in \mathfrak{S}$ it will be convenient to extend ϵ to a function $\mathbf{Z}_{>0} \to \{-1, 0, 1\}$ (denoted again by ϵ) by setting $\epsilon(i) = -1$ if i is odd or $\mu_i(c_*) = 0$.

Now let γ, γ' be as in 1.3. We write $\gamma = \gamma_{c_*,\epsilon}, \gamma' = \gamma_{c'_*,\epsilon'}$ with $(c_*, \epsilon), (c'_*, \epsilon') \in \mathfrak{S}$. Let $g \in \gamma_{c'_*,\epsilon'}$ be such that $gV_* = V'_*$ and let $N = g - 1 : V \to V$. To prove that γ is contained in the closure of γ' in G it is enough to show that

(a) $c_* \le c'_*$

and that for any $i \ge 1$, (b),(c) below hold:

(b) $\sum_{j\in[1,i]} c_j^* - \max(\epsilon(i),0) \ge \sum_{j\in[1,i]} {c'}_j^* - \max(\epsilon'(i),0)$;

(c) if $\sum_{j\in[1,i]} c_j^* = \sum_{j\in[1,i]} {c'}_j^*$ and $c_{i+1}^* - {c'}_{i+1}^*$ is odd then $\epsilon'(i) \ne 0$.

(See [S, II,8.2].) From the definition we see that $c_i = 2p_i$ for $i \in [1,\sigma]$, $c_i = 0$ for $i > \sigma$ and from [L1, 4.6] we see that $\epsilon(i) = 1$ for all $i \in \{2,4,6,\dots\}$ such that $\mu_i(c_*) > 0$.

Now (a) follows from [L1, 3.5(a)]. Indeed in *loc.cit.* it is shown that for any $i \ge 1$ we have $\dim N^iV \ge \Lambda_i$ where

$$\Lambda_i = \sum_{j\ge 1; i\le c_j} (c_j - i) = \sum_{j\in[1,c_i^*]} (c_j - i).$$

We have $\dim N^iV = \sum_{j\ge1;i\le c'_j}(c'_j - i) = \sum_{j\in[1,{c'}_i^*]}(c'_j - i)$ hence by 1.4(d) the inequality $\dim N^iV \ge \Lambda_i$ is the same as the inequality $\sum_{j\in[1,i]} c_j^* \ge \sum_{j\in[1,i]} {c'}_j^*$.

Note also that, by 1.4(d),

(d) *we have* $\sum_{j\in[1,i]} c_j^* = \sum_{j\in[1,i]} {c'}_j^*$ *iff* $\dim N^iV = \Lambda_i$.

1.6. Let $i \ge 1$. We show:

(a) *If* $\mu_i(c_*) > 0$ *and* $\sum_{j\in[1,i]} c_j^* = \sum_{j\in[1,i]} {c'}_j^*$ *then* $\epsilon'(i) = 1$.

By 1.4(e) we have $\mu_i(c'_*) > 0$. Since $\mu_i(c_*) > 0$ we see that $i = 2p_d$ for some $d \in [1,\sigma]$. If $\mu_i(c'_*)$ is odd then $\epsilon'(i) = 1$ (by definition, since i is even). Thus we may assume that $\mu_i(c'_*) \in \{2,4,6,\dots\}$. From our assumption we have that $\dim N^iV = \Lambda_i$ (see 1.5(d)).

Let $v_1, v_2, \dots, v_\sigma$ be vectors in V attached to V_*, V'_*, g as in [L1, 3.3]. For $r \in [1,\sigma]$ let W_r, W'_r be as in [L1, 3.4]; we set $W_0 = 0$, $W'_0 = V$. From [L1, 3.5(b)] we see that $N^iW'_{d-1} = 0$ at least if $d \ge 2$; but the same clearly holds if $d = 1$. We have $v_d \in W'_{d-1}$ hence $N^{2p_d}v_d = 0$ and

$$(N^{2p_d-1}(v_d), v_d) = (N^{p_d}v_d, N^{p_d-1}v_d) = ((g-1)^{p_d}v_d, (g-1)^{p_d-1}v_d) =$$
$$(g^{p_d}v_d, v_d) = 1.$$

(We have used that $(v_d, g^kv_d) = 0$ for $k \in [-p_d+1, p_d-1]$ and $(v_d, g^{p_d}v_d) = 1$, see [L1, 3.3(iii)].) Thus $\epsilon'(i) = 1$. This proves (a).

1.7. We prove 1.5(b). It is enough to show that, if $\epsilon(i) = 1$ and $\epsilon'(i) \le 0$ then $\sum_{j\in[1,i]} c_j^* \ge \sum_{j\in[1,i]} {c'}_j^* + 1$. Assume this is not so. Then using 1.5(a) we have $\sum_{j\in[1,i]} c_j^* = \sum_{j\in[1,i]} {c'}_j^*$. Since $\epsilon(i) = 1$ we have $\mu_i(c_*) > 0$; using 1.6(a) we see that $\epsilon'(i) = 1$, a contradiction. Thus 1.5(b) holds.

1.8. We prove 1.5(c). If i is odd then $\epsilon'(i) = -1$, as required. Thus we may assume that i is even. Using 1.5(a) and 1.4(e) we see that $c_i^* \le {c'}_i^*$.

Assume first that $c_i^* = {c'}_i^*$. From $\mu_i(c_*) = c_i^* - c_{i+1}^*$, $\mu_i(c'_*) = {c'}_i^* - {c'}_{i+1}^*$ we deduce that $\mu_i(c_*) - \mu_i(c'_*) = {c'}_{i+1}^* - c_{i+1}^*$ is odd. If $\mu_i(c'_*)$ is odd we have

$\epsilon'(i) = 1$ (since i is even); thus we have $\epsilon'(i) \neq 0$, as required. If $\mu_i(c'_*) = 0$ we have $\epsilon'(i) = -1$; thus we have $\epsilon'(i) \neq 0$, as required. If $\mu_i(c'_*) \in \{2, 4, 6, \dots\}$ then $\mu_i(c_*)$ is odd so that $\mu_i(c_*) > 0$ and then 1.6(a) shows that $\epsilon'(i) = 1$; thus we have $\epsilon'(i) \neq 0$, as required.

Assume next that $c^*_i < c'^*_i$. By 1.5(a) we have $\sum_{j\in[1,i+1]} c^*_j \geq \sum_{j\in[1,i+1]} c'^*_j$; using the assumption of 1.5(c) we deduce that $c^*_{i+1} \geq c'^*_{i+1}$. Combining this with $c^*_i < c'^*_i$ we deduce $c^*_i - c^*_{i+1} < c'^*_i - c'^*_{i+1}$ that is, $\mu_i(c_*) < \mu_i(c'_*)$. It follows that $\mu_i(c'_*) > 0$. If $\mu_i(c_*) > 0$ then by 1.6(a) we have $\epsilon'(i) = 1$; thus we have $\epsilon'(i) \neq 0$, as required. Thus we can assume that $\mu_i(c_*) = 0$. We then have $c^*_i = c^*_{i+1}$ and we set $\delta = c^*_i = c^*_{i+1}$. As we have seen earlier, we have $c^*_{i+1} \geq c'^*_{i+1}$; using this and the assumption of 1.5(c) we see that $c^*_{i+1} - c'^*_{i+1} = 2a + 1$ where $a \in \mathbf{N}$. It follows that $c'^*_{i+1} = \delta - (2a + 1)$. In particular we have $\delta \geq 2a + 1 > 0$.

If $k \in [0, 2a]$ we have $c'_{\delta-k} = i$. (Indeed, assume that $i + 1 \leq c'_{\delta-k}$; then by 1.4(b) we have $\delta - k \leq c'^*_{i+1} = \delta - (2a + 1)$ hence $k \geq 2a + 1$, a contradiction. Thus $c'_{\delta-k} \leq i$. On the other hand, $\delta = c^*_i < c'^*_i$ implies by 1.4(b) that $i \leq c'_\delta$. Thus $c'_{\delta-k} \leq i \leq c'_\delta \leq c'_{\delta-k}$ hence $c'_{\delta-k} = i$.)

Using 1.4(b) and $c'^*_{i+1} = \delta - (2a + 1)$ we see that $c'_{\delta-(2a+1)} \geq i + 1$ (assuming that $\delta - (2a + 1) > 0$). Thus the sequence $c'_1, c'_2, \dots, c'_\delta$ contains exactly $2a + 1$ terms equal to i, namely $c'_{\delta-2a}, \dots, c'_{\delta-1}, c'_\delta$.

We have $i > c_{\delta+1}$. (If $i \leq c_{\delta+1}$ then from 1.4(b) we would get $\delta + 1 \leq c^*_i = \delta$, a contradiction.)

Since $\delta > 0$, from $c^*_i = \delta$ we deduce that $i \leq c_\delta$ (see 1.4(b)); since $\mu_i(c_*) = 0$ we have $c_\delta \neq i$ hence $c_\delta > i$. From the assumption of 1.5(c) we see that $\dim N^i V = \Lambda_i$ (see 1.5(d)). Using this and $c_\delta > i > c_{\delta+1}$ we see that [L1, 3.5] is applicable and gives that $V = W_\delta \oplus W_\delta^\perp$ and W_δ, $W_\delta^\perp$ are g-stable; moreover, $g : W_\delta \to W_\delta$ has exactly δ Jordan blocks and each one has size $\geq i$ and $g : W_\delta^\perp \to W_\delta^\perp$ has only Jordan blocks of size $\leq i$. Since the δ largest numbers in the sequence $c'_1, c'_2, \dots$ are $c'_1, c'_2, \dots, c'_\delta$ we see that the sizes of the Jordan blocks of $g : W_\delta \to W_\delta$ are $c'_1, c'_2, \dots, c'_\delta$. Since the last sequence contains an odd number ($= 2a + 1$) of terms equal to i we see that $\epsilon_{g|_{W_d}}(i) = 1$. (Note that $(,)$ is a nondegenerate symplectic form on W_d hence $\epsilon_{g|_{W_d}}(i)$ is defined as in 1.5.) Hence there exists $z \in W_d$ such that $N^i z = 0$ and $(z, N^{i-1}z) = 1$. This implies that $\epsilon_g(i) = 1$ that is $\epsilon'(i) = 1$. This completes the proof of 1.5(c) and also completes the proof of Theorem 1.3 when $Q = 0$.

1.9. In this subsection we assume that $Q \neq 0$. Let γ, γ' be as in 1.3. We denote by γ_1, γ'_1 the $Is(V)$-conjugacy class containing γ, γ' respectively; let γ_2, γ'_2 be the $Sp(V)$-conjugacy class containing γ_1, γ'_1 respectively. Note that Theorem 1.3 is applicable to γ_2, γ'_2 instead of γ, γ' and with G replaced by the larger group $Sp(V)$. Thus we have that γ_2 is contained in the closure of γ'_2 in $Sp(V)$ and then, using [S, II,8.2], we see that γ_1 is contained in the closure of γ'_1 in $Is(V)$. We have $\gamma_1 = \gamma$ (see [S, I,2.6]). If $\gamma'_1 = \gamma'$ it follows that γ is contained in the closure of γ' in G, as required. If $\gamma'_1 \neq \gamma'$ then $\gamma'_1 = \gamma' \sqcup \gamma''$ where $\gamma'' = r\gamma' r^{-1}$ (r is a fixed element

in $Is(V) - G$). We see that either γ is contained in the closure of γ' or in the closure of $r\gamma' r^{-1}$. In the last case we have that $r^{-1}\gamma r$ is contained in the closure of γ'. But $r^{-1}\gamma r = \gamma$ so that in any case γ is contained in the closure of γ'. This completes the proof of Theorem 1.3 when $Q \neq 0$.

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Department of Mathematics, M.I.T., Cambridge, MA 02139; Department of Mathematics, Northwestern University, Evanston, IL 60208